Automatic Photo Reordering in a Simultaneous Bundle Adjustment

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Abstract

A method of automatic photo reordering for efficient and economic simultaneous bundle adjustments regardless of the configuration of the block of photos is presented. After discussing the manual photo ordering process, an automatic photo reordering approach for bandwidth and/or profile reduction of the coefficient matrix of the reduced normal equations is presented. The potential of the bundle adjustment is enhanced by the implementation of automatic photo connectivity constructions and a heuristic reordering algorithm based on graph theory. The automatic photo reordering program is equally applicable to regular and irregular blocks of photos for fast and near optimal analytical solutions.

Introduction to the Photogrammetric Bundle Adjustment

The effective correction of all the measurements in a photogrammetric mapping problem forms one of the strong foundations for achieving high accuracy in analytical aerotriangulation. A simultaneous least-squares adjustment of photogrammetric blocks (also known as "bundle" adjustment), applying the collinearity condition and observation equations for the exterior orientation parameters of the photos and ground control points, provides the most accurate results in a single solution. The mathematical model and solution of a standard photogrammetric block bundle adjustment was presented by Brown *et al.* (1964) and Brown (1968), and was extended by Wong and Elphingstone (1972) for additional ground observations. The system of photogrammetric equations, in matrix notation, is as follows (see Wong (1980, pp. 88-101) for detailed information):

$$\begin{bmatrix} V \\ \dot{V} \\ \ddot{V} \end{bmatrix} + \begin{bmatrix} \dot{B} & \ddot{B} \\ -I & O \\ O & -I \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \dot{C} \\ \ddot{C} \end{bmatrix} \text{ or } \bar{V} + \bar{B}\,\Delta = \bar{C} \qquad (1)$$

The set of normal equation is

$$(\bar{B}^T \bar{W} \bar{B})\,\Delta = \bar{B}^T \bar{W} \bar{C} \quad \text{or} \quad N\,\Delta = K \qquad (2)$$

which can be further partitioned and expressed as Equations 3 and 4: i.e.,

$$\ddot{\Delta} = \ddot{N}^{-1} (\ddot{K} - \bar{N}^T\ \dot{\Delta}) \qquad (3)$$

$$(\dot{N} - \bar{N}\,\ddot{N}^{-1}\,\bar{N}^T)\,\dot{\Delta} = (\dot{K} - \bar{N}\,\ddot{N}^{-1}\,\ddot{K}) \quad \text{or} \quad S\,\dot{\Delta} = E \qquad (4)$$

Equation 4 is often referred to as the reduced normal equation. The structure and size of the coefficient matrix **S** depends on a number of parameters, including the geometric configuration of the block and the photo ordering scheme of the model, which dominates the structure of **S**. For a block of *m* photos, the direct solution of the reduced normal equation involves inverting the **S** matrix, which is 6*m* by 6*m* in size. Fortunately, the sparse symmetric banded structure of the **S** matrix enables efficient solution of the reduced normal equations by applying storage minimization techniques such as Cholesky factorization and recursive partitioning. The efficiency of the solution depends primarily on how the algorithm exploits the sparsity of the **S** matrix, in terms of the ordering of photos, for minimal bandwidth and/or profile.

The remainder of this paper is organized as follows: Manual approaches for photo ordering to reduce the bandwidth and/or profile of the **S** matrix of the reduced normal equations are first described; a method of automatic photo reordering for efficient and economic solutions in simultaneous bundle adjustment of photogrammetric problems regardless of the configuration and shape of the block of photos in the problems is then presented; the proposed method automatically constructs graph structures of photos, and employs a heuristic algorithm for reducing the bandwidth and/or profile of the coefficient matrix; and finally, the advantages of the proposed approach in enhancing the simultaneous bundle adjustment are presented.

Manual Photo Ordering

Manual ordering of photos usually takes place before making the measurements of photo coordinates of image points, or during the project planning stage. In the process, all measurements are ordered to reduce the core storage requirements and computational time for a bundle adjustment. In aerotriangulation, each photo overlaps with only a few photos in the block. Thus, by appropriate ordering of the photos in advance, the bandwidth and/or profile of the **S** matrix can be reduced for efficient storage and matrix manipulations in a bundle adjustment solution.

For a regular block of photos with standard control extension, the structure of the **S** matrix varies depending upon the ordering of photos and the configuration of the block. Variables include the number of strips (*s*), number of photos per strip (*r*), and percentages of endlap and sidelap. For example, for a regular block of photos having 60 percent endlap and 20 percent sidelap, the bandwidth of the **S** matrix is

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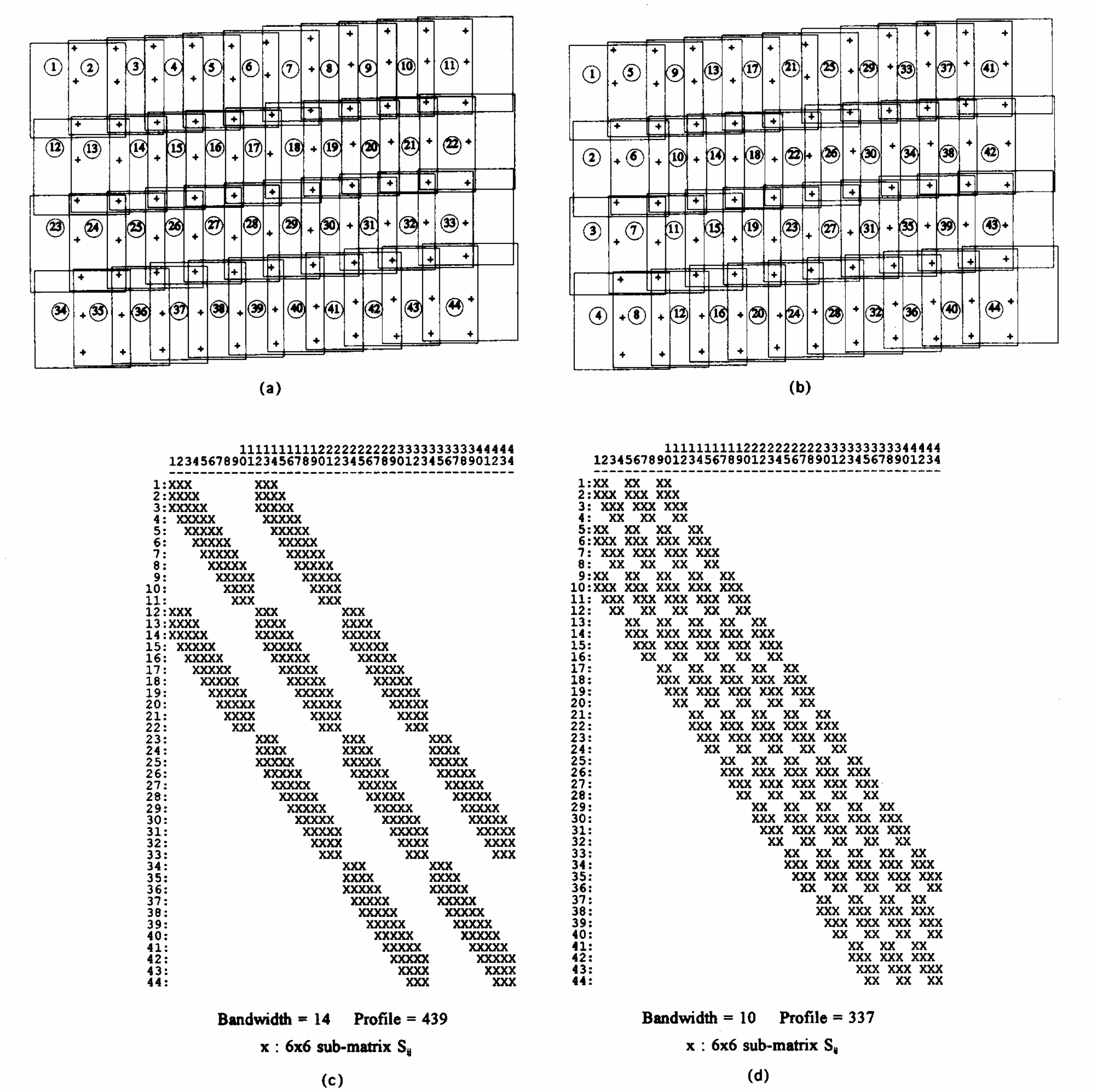


Figure 1. (a) Regular block of photos with down-strip numbering. (b) Regular block of photos with cross-strip numbering. (c) **S**-matrix structures of the 4 by 11 block of photos with down-strip numbering. (d) **S**-matrix structures of the 4 by 11 block of photos with cross-strip numbering.

$6(r + 3)$ for down-strip numbering, and $6(2s + 2)$ for cross-strip numbering (Wong, 1980). Figure 1 depicts a 4 by 11 block of photos and corresponding structures of the **S** matrices for down-strip numbering and cross-strip ordering. Both the bandwidth and profile of the **S** matrix are computed in terms of basic 6 by 6 sub-matrices to show the benefit in bandwidth/profile reduction. Thus, the manual ordering scheme can be easily determined in advance by choosing the one which yields a small bandwidth of the **S** matrix.

However, not every photogrammetric block is ideally regular in shape. Irregularity of photogrammetric blocks may occur due to irregular project boundaries, alternation and shift of strips, irregular scale and orientation of photographs, and deviation of flight parameters such as crab, drift, tilt, inclination, and others. For these irregular blocks, the ordering of photos generally becomes a "trial and error" process, searching for a scheme that yields the **S** matrix having a bandwidth and/or profile as small as possible. In the process, one can generate all possible orderings of photos, compute the resulting bandwidth and profile of the **S** matrix for each, and then take the minimum. However, it is often difficult for a "trial and error" manual ordering process to achieve an op-

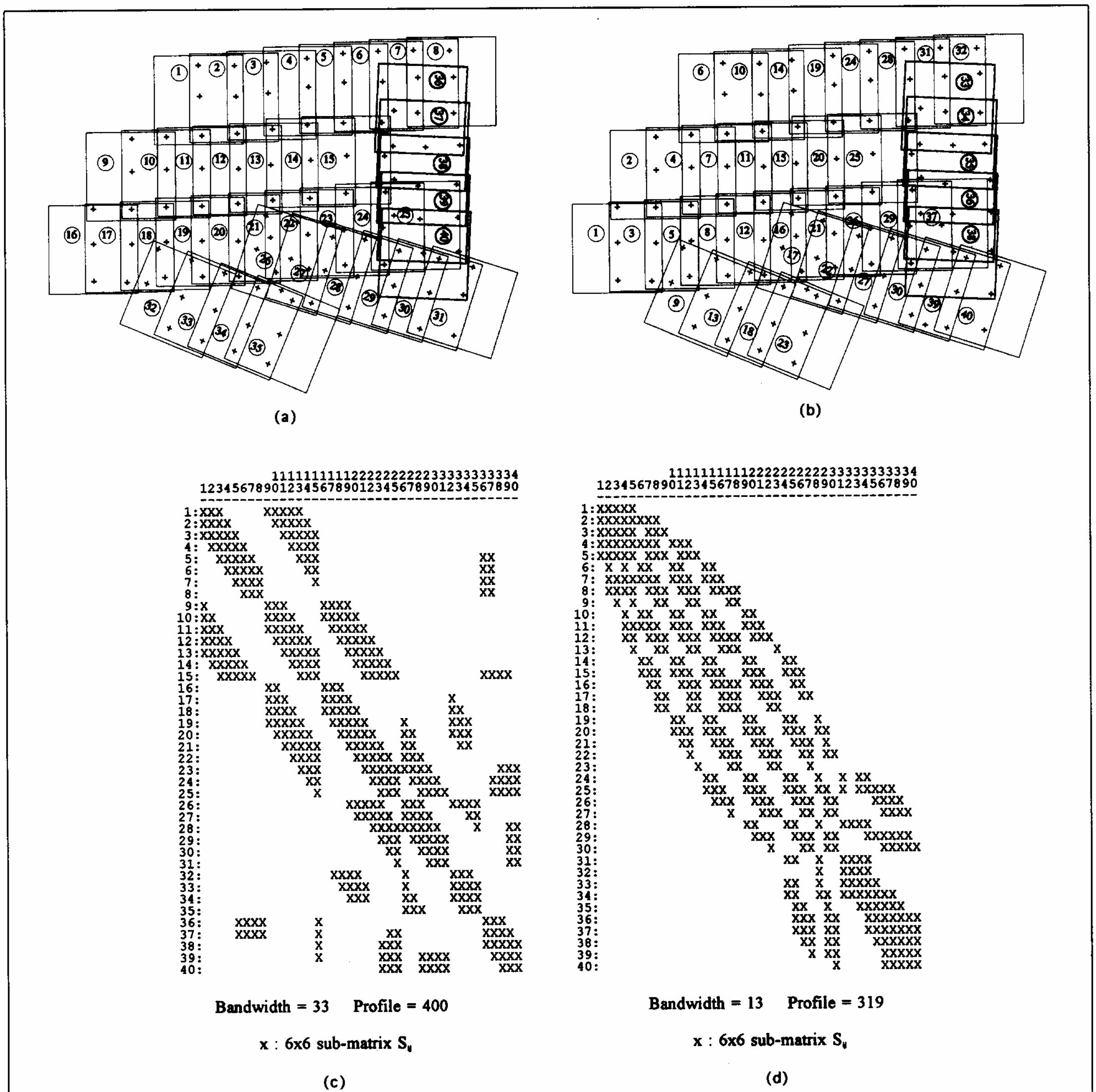


Figure 2. (a) Irregular block of photos with down-strip numbering. (b) Irregular block of photos with cross-strip numbering. (c) **S**-matrix structures of the irregular block of photos with down-strip numbering. (d) **S**-matrix structures of the irregular block of photos with cross-strip numbering.

timal solution of the **S** matrix for an irregular photogrammetric block in a reasonable length of time without wasting resources.

Though one can follow the procedures in ordering photos of a regular block, the resulting structure of the **S** matrix may not have minimal bandwidth and/or profile. For example, Figure 2 illustrates an irregular block of 40 photos and shows structures of the **S** matrix for down-strip and cross-strip ordering schema. The significant reductions in both bandwidth and profile of the **S** matrix imply that the cross-strip ordering scheme delivers a preliminarily acceptable "trial and error" solution. But efficient and economic solutions can be achieved by applying automatic photo reordering approaches without consideration of the photo numbering process.

Automatic Photo Reordering

The photo ordering process can be automated by applying computerized programs that take advantage of "automatic model construction" and heuristic algorithms for sparse matrix bandwidth and/or profile reduction. Automatic model construction involves automatically identifying the connec-

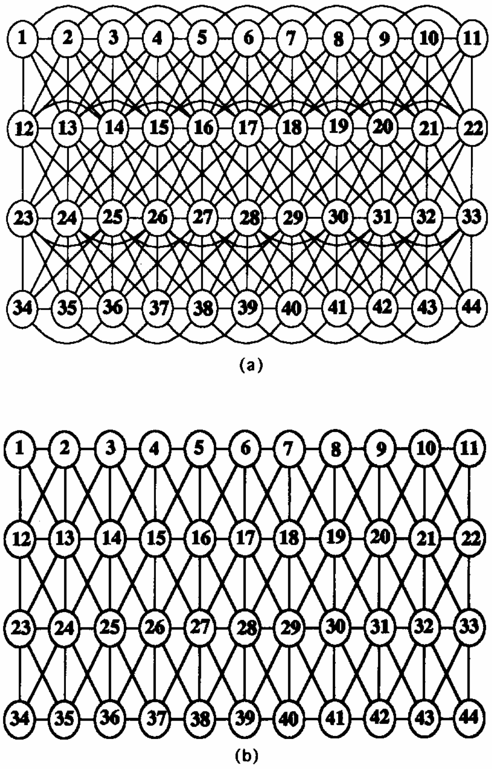


Figure 3. Graph structures of the 4 by 11 regular block of photos in Figure 1a. (a) Complete graph structures of photos. (b) Reduced graph structures of photos.

tivity and overlapping nature of photographs by analysis of common image points. By further applying heuristic algorithms, the bandwidth and/or profile of the **S** matrix is reduced using the photo connectivity information. The result is an automatic photo reordering scheme for application in a simultaneous bundle adjustment. Thus, the ordering of photos can be originally arbitrary and eliminated from the premeasurement tasks in analytical aerotriangulation.

Automatic Model Construction

The structural formation of the **S** matrix can be explored by identifying the connectivity and overlapping of photographs through an analysis of shared common image points. The topological concept of connectivity of photographs can be represented as graphs. A graph in this context is defined as a collection of nodes (i.e., photos) and links which connect pairs of nodes. Though there exist various kinds of graphs, this application deals only with undirected graphs because of the characteristics of photogrammetric problems.

Two photos (nodes) i and j are connected by a link in the graph structure if there exist image points common in both photos. Thus, the 6 by 6 sub-matrix $\mathbf{S}_{ij}$ consists of the contributions of the collinearity equations from all the image points common in both photos i and j. On the other hand, if photos i and j do not share any common image points, no link will appear between them in the graph structure, i.e., $\mathbf{S}_{ij} = \mathbf{0}$. Thus, the complete graph structure of photos can be graphically drawn upon the connectivity of photos. For example, Figures 3a and 4a show the complete graph structures of the 4 by 11 regular block of photos in Figure 1a and the irregular block of photos in Figure 2a, respectively.

The graph structure of photos can be maintained in terms of adjacency lists of photos in a computer program. Meanwhile, automatic construction of the graph structure of photos is facilitated by the use of tabular data structures and the concept of primary key found in relational database management systems. Table 1 illustrates the relational data structures for cameras, photos, and object and image points which are assigned unique identifiers as the primary key in all data structures. In this data structure, note that image points on different photos of the same object point should have an identical identifier as the object point. Whenever an image point is found on two photos, corresponding photo identifiers are appended to the adjacency lists of the two photos, and values of the *nadj*'s updated. In this manner, corresponding graphical links among photos in the graph structure are built.

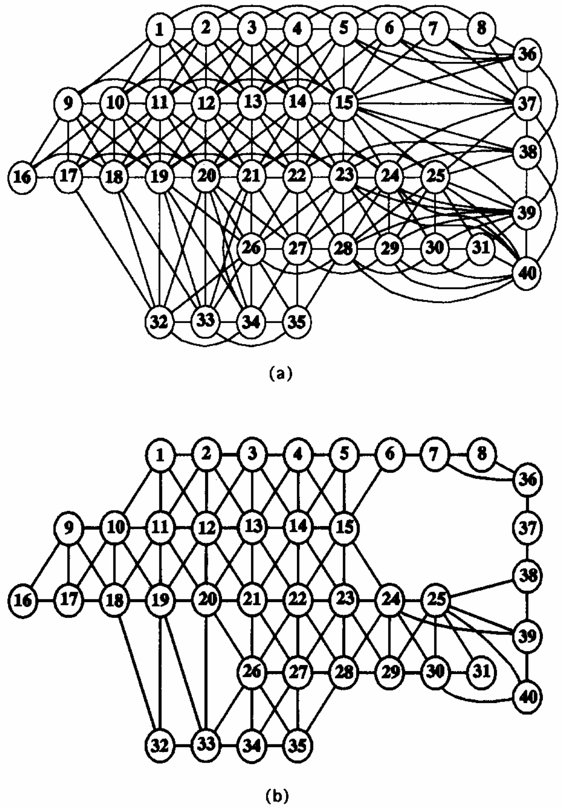


Figure 4. Graph structures of the irregular block of photos in Figure 2a. (a) Complete graph structures of photos. (b) Reduced graph structures of photos.

Table 1. Relational Data Structures for Camera, Photo, and Object and Image Points in the Simultaneous Bundle Adjustment Module of the Analytical Photogrammetry Aerotriangulation System (APAS) (Tsai, 1992).

Structure	Element	Key	Data Type	Description
CAMERA	CameraID	✓	string^s	Camera identifier (^{s} array of characters)
	Focal		float	Focal length of the camera
	CrFocal		float	Calibration to the focal length
	PPxy[2]		float	(x_0,y_0) coordinates of the principal point
	CrPPxy[2]		float	Calibrations to (x_0,y_0) of the principal point
PHOTO	PhotoID	✓	string	Photo identifier
	CameraID		string	Camera by which the photo was taken
	XYZ[3]		float	(X,Y,Z) coordinates of the exposure center
	CrXYZ[3]		float	Corrections to (X,Y,Z) of the exposure center
	RsXYZ[3]		float	Residuals of (X,Y,Z) of the exposure center
	WPK[3]		float	(ω,ϕ,κ) rotations of the exposure center
	CrWPK[3]		float	Corrections to (ω,ϕ,κ) of the exposure center
	RsWPK[3]		float	Residuals of (ω,ϕ,κ) of the exposure center
	M[9]		float	Rotation matrix **M** of the exposure center
	nadj		int	Total number of connected (overlapped) photos
	xadj		int	Reduced number of connected photos
	adjlist		int*	Pointer to an integer array (the adjacency list) which holds *nadj* indices of connected photos
OBJECT	ObjectID	✓	string	Object point identifier
	ObXYZ[3]		float	(X,Y,Z) coordinates of the object point
	CrXYZ[3]		float	Corrections to (X,Y,Z) of the object point
	RsXYZ[3]		float	Residuals of (X,Y,Z) of the object point
IMAGE	ObjectID	✓	string	Object point identifier for the image point
	PhotoID		string	Photo on which the image point was measured
	ImXY[2]		float	(x,y) photo coordinates of the image point
	RsXY[2]		float	Residuals of (x,y) of the image point

The adjacency lists of photos represent the complete graph structure of photos in a photogrammetric block, and can be used in an automatic photo reordering algorithm to reduce the bandwidth and/or profile of the **S** matrix. However, as shown in Figures 3a and 4a, the complete graph structures are too complex to achieve desirable ordering of photos for bandwidth and/or profile minimization when applying the automatic photo reordering algorithm. Although the procedure does reduce the bandwidth and/or profile of the **S** matrix, an optimal solution does not result. Thus, minor modifications of the graph structure and the adjacency lists are necessary for a near optimal solution.

Modification of the complete graph structure takes place by deleting the links which connect photos that (1) are not stereo pairs in a strip but share common image points, e.g., photos 1 and 3, 2 and 4, etc.; and (2) lie in adjacent strips but share only one image point at opposite corners of the photos, e.g., photos 1 and 14, 14 and 23, etc. These modifications form the reduced graph structure of photos, e.g., Figure 3b for the 4 by 11 regular block in Figure 1a, and Figure 4b for the irregular block in Figure 2a. In the computer program, these photos are detected by computing relative positions between them from initial approximations of the coordinates of photo exposure centers.

Modification of the adjacency lists does not remove any elements from the lists of corresponding photos whose connections were deleted in the reduced graph structure. Instead, elements of the adjacency lists of corresponding photos are swapped in position, and values of the *xadj*'s of the photos are updated while the *nadj*'s remain unchanged. Together with the adjacency lists of photos, the *nadj*'s and the *xadj*'s represent the complete and the reduced graph structures, respectively. Both digital representations of graph structures of photos are used in the automatic reordering algorithm described in the next section.

S-Matrix Bandwidth and Profile Reduction

In general, the objective of photo reordering in the photogrammetric bundle adjustment is to form the coefficient matrix **S** so that the non-zero elements in **S** are clustered near the main diagonal for minimum storage requirements and computational effort. This often results in reducing either the bandwidth or the profile, or both, of the **S** matrix. There are many algorithms for reordering equations or unknowns to preserve sparsity and obtain a minimal storage for sparse matrices. For examples, see Alway and Martin (1965), Cuthill and McKee (1969), George (1971), Snay (1976), and Gibbs *et al.* (1976). George and Liu (1981) and Duff *et al.* (1986) gave the theoretical background and practical examples of handling large systems of sparse linear equations.

In this presentation, the graph-based heuristic algorithm, called the GPS algorithm because it was developed by Gibbs *et al.* (1976), has been adopted. It reduces the bandwidth and profile of a sparse matrix in significantly reduced time. The GPS algorithm is an improved version of the reverse Cuthill-McKee algorithm (George, 1971). The primary improvement consists in making a better selection of a starting node over a more general type of level structure. Based on the use of graph structures, the GPS algorithm proceeds the following procedures (Gibbs *et al.*, 1976):

(1) Find endpoints of a pseudo-diameter, i.e., a pair of nodes that are at nearly maximal distance apart in the graph structure:
(A) Start from the vertex *v*, which has a minimal degree (the

TABLE 2. LEVEL STRUCTURES OF THE 4 BY 11 REGULAR BLOCK OF PHOTOS IN FIGURE 1A.

Level Structure	Level	Using Complete Graph Structures (Figure 3a)	Using Reduced Graph Structures (Figure 3b)
Rooted **V**	1	44	44
	2	32 33 43 31 42	32 33 43
	3	20 21 22 19 30 41 18 29 40	20 21 22 31 42
	4	8 9 10 7 11 6 17 28 39 5 16 27 38	8 9 10 19 30 11 41
	5	4 15 26 37 3 14 25 36	7 18 29 40
	6	2 13 24 35 1 12 23 34	6 17 28 39
	7		5 16 27 38
	8		4 15 26 37
	9		3 14 25 36
	10		2 13 24 35
	11		1 12 23 34
Rooted **U**	1	12	12
	2	1 2 13 23 24 3 14 25	1 2 13 23 24
	3	4 15 26 34 35 36 37 5 16 27 38	3 14 25 34 35 36
	4	6 17 28 39 7 18 29 40	4 15 26 37
	5	8 19 30 41 9 20 31 42	5 16 27 38
	6	10 21 32 43 11 22 33 44	6 17 28 39
	7		7 18 29 40
	8		8 19 30 41
	9		9 20 31 42
	10		10 21 32 43
	11		11 22 33 44
Minimized	1	44 11 43 22 10 33 21 32	44 11 22 33
	2	9 41 42 8 20 30 31 19	10 43 21 32
	3	7 39 40 6 28 17 29 18	42 9 31 20
	4	34 38 37 35 5 36 4 27 16 26 15	41 8 30 19
	5	1 23 24 2 25 13 3 14	40 7 29 18
	6	12	39 6 28 17
	7		38 5 27 16
	8		37 4 26 15
	9		36 3 25 14
	10		35 2 24 13
	11		34 23 1 12
Reordered	1	44 43 33 32 22 21 11 10	1 12 23 34
	2	42 31 41 30 20 19 9 8	2 13 24 35
	3	40 29 18 39 28 17 7 6	3 14 25 36
	4	38 27 16 5 37 26 15 4 36 35 34	4 15 26 37
	5	25 14 3 24 13 2 23 1	5 16 27 38
	6	12	6 17 28 39
	7		7 18 29 40
	8		8 19 30 41
	9		9 20 31 42
	10		10 21 32 43
	11		11 22 33 44

degree of a vertex is the number of vertices adjacent to it), and generate the level structure **V** rooted at v;

(B) For any vertex t in the last level of **V**, generate the level structure **T** rooted at t. If the depth of **T**, i.e., the number of levels of **T**, is smaller than the depth of **V**, then assign t to v and **T** to **V**, and repeat the step. The step ends with a rooted level structure **V** that has a minimal depth;

(C) Generate the level structure **U** rooted at vertex u in the last level of **V** which has the smallest width (the width of a level structure is the maximum number of vertices in each level of the level structure).

(2) Minimize level width by combining the two rooted level structures **V** and **U** into a general level structure **G**:

(A) Associate each vertex an index pair (i, j), where i is its index of the level in **V** and $k+1-j$ is its index of the level in **U**, where k is the depth of **V** and **U**;

(B) Assign vertices with index pair (i, i) to level $\mathbf{G}_i$ of **G**, the graph then consists of a set of one or more disjoint connected components $C_1, C_2, \ldots, C_t$ ordered so that $n(C_1) > n(C_2) > \ldots > n(C_t)$, where $n(C_i)$ is the number of vertices in the component C_i;

(C) For each connected component C_i, i = 1, 2, ..., t, do:

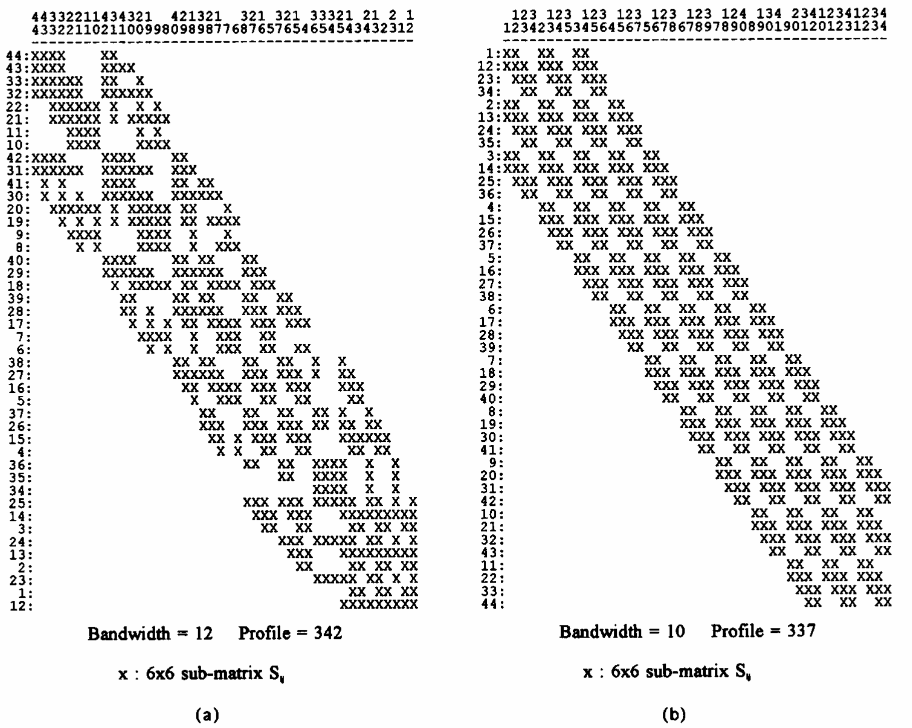


Figure 5. (a) Reordered **S**-matrix structures using the complete graph structures of the 4 by 11 block of photos. (b) Reordered **S**-matrix structures using the reduced graph structures of the 4 by 11 block of photos.

(i) Compute (g_1, g_2, ..., g_k), where g_j is the number of vertices in level $\mathbf{G}_j$;
(ii) Compute (v_1, v_2, ..., v_k) and (u_1, u_2, ..., u_k), where v_j = g_j + (the number of vertices which would be placed in level $\mathbf{G}_j$ if the first index is used), and u_j = g_j + (the number of vertices which would be placed in level $\mathbf{G}_j$ if the second index is used);
(iii) Find v_0 = max (v_1, v_2, ..., v_k) and u_0 = max (u_1, u_2, ..., u_k), then
  (a) if $v_0 < u_0$, place vertices into **G** using the first index;
  (b) if $v_0 > u_0$, place vertices into **G** using the second index;
  (c) if $v_0 = u_0$, place vertices into **G** using the index of the rooted level structure with smaller width. If the width are equal, use the first index.

(3) Renumber the nodes, level by level, of the general level structure **G**:
  (A) If the degree of u is less than the degree of v, then interchange u and v and reverse the general level structure **G**;
  (B) Assign consecutive positive integers to the vertices of level $\mathbf{G}_1$:
    (i) Assign the number 1 to the vertex v;
    (ii) Let w be the lowest numbered vertex in level $\mathbf{G}_1$ which has unnumbered vertices in level $\mathbf{G}_1$ adjacent to it. Number the vertices of level $\mathbf{G}_1$ adjacent to w, in order of increasing degree. Repeat the step until all vertices of level $\mathbf{G}_1$ adjacent to numbered vertices are numbered;
    (iii) If any unnumbered vertices in level $\mathbf{G}_1$, number the one of minimal degree, then go to Step (ii). Otherwise proceed to Step (C).
  (C) Number the vertices of level $\mathbf{G}_i$, i = 2, 3, ..., k, as follows:
    (i) Let w be the lowest numbered vertex of $\mathbf{G}_{i-1}$ that has unnumbered vertices of $\mathbf{G}_i$ adjacent to it. Number the vertices of $\mathbf{G}_i$ adjacent to w in order of increasing degree. Repeat this step until all vertices of $\mathbf{G}_i$ adjacent to vertices of $\mathbf{G}_{i-1}$ are numbered;
    (ii) Repeat Step (Bii) and (Biii), replacing $\mathbf{G}_1$ with $\mathbf{G}_i$.
  (D) Reverse the numbering if
    (i) Step (A) interchanged vertices v and u and sub-algorithm 2 selected the second indices of the index pairs for component C_1; or
    (ii) Step (A) did not interchange vertices v and u and sub-algorithm 2 selected the first indices of the index pairs for component C_1.

The GPS algorithm was implemented in the simultaneous bundle adjustment module of the Analytical Photogrammetry Aerotriangulation System (APAS) (Tsai, 1992) using the C computer programming language. Instead of actually renumbering the photo identifiers, the GPS renumbering sub-algorithm was modified to generate a permutation array for the new order of photos. The adjacency lists of photos, as described in the section on Automatic Model Construction, are used as input graph structures, including both the complete and the reduced graph structures, to the GPS reordering program, selecting the one which yields a small bandwidth and/or profile of the **S** matrix. The resulting permutation array is then used as the photo reordering mechanism for all follow-

TABLE 3. LEVEL STRUCTURES OF THE IRREGULAR BLOCK OF THE 40 PHOTOS IN FIGURE 2A.

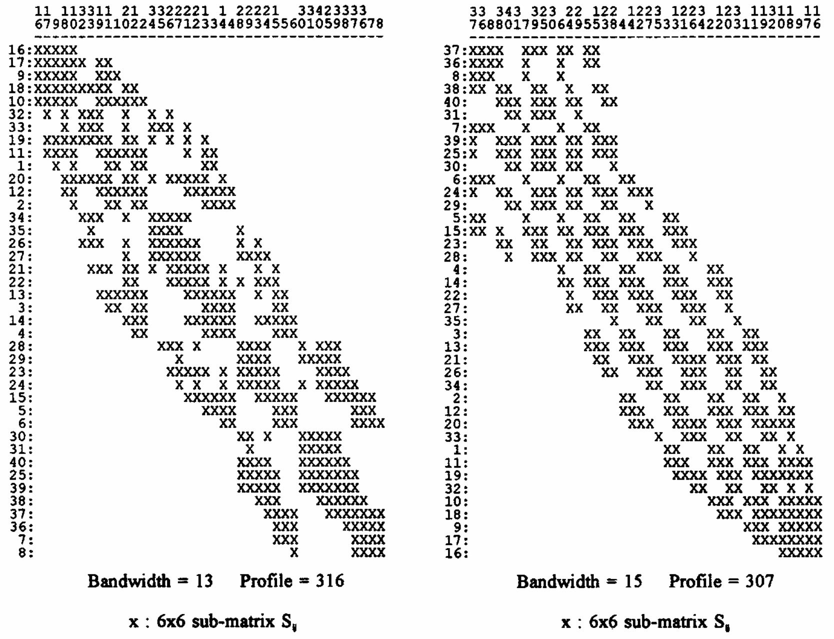
Level Structure	Level	Using Complete Graph Structures (Figure 4a)	Using Reduced Graph Structures (Figure 4b)
Rooted **V**	1	16	16
	2	9 17 10 18	9 17
	3	1 11 19 32 2 12 20 33	10 18
	4	3 13 21 26 34 4 14 22 27 35	1 11 19 32
	5	5 15 23 28 6 24 29	2 12 20 33
	6	7 36 37 25 38 39 40 30 8 31	3 13 21 26 34
	7		4 14 22 27 35
	8		5 15 23 28
	9		6 24 29
	10		7 25 30 39
	11		8 36 31 38 40
	12		37
Rooted **U**	1	37	37
	2	36 38 5 6 7 8 15 24 25 39	36 38
	3	23 40 4 14 3 13 22 21 28 29 30 27 31	7 8 25 39
	4	26 2 12 20 1 11 19 33 34 35	6 24 29 30 31 40
	5	32 10 18 9 17	5 15 23 28
	6	16	4 14 22 27 35
	7		3 13 21 26 34
	8		2 12 20 33
	9		1 11 19 32
	10		10 18
	11		9 17
	12		16
Minimized	1	16	16
	2	17 18 9 10	17 9
	3	32 33 2 1 12 11 19 20	18 10
	4	35 34 4 22 27 3 14 26 13 21	32 1 19 11
	5	6 29 5 28 24 23 15	33 2 20 12
	6	8 31 30 40 38 36 7 39 37 25	34 3 26 13 21
	7		35 27 4 14 22
	8		28 5 23 15
	9		6 29 24
	10		30 39 7 25
	11		31 40 8 38 36
	12		37
Reordered	1	16	37
	2	17 9 18 10	36 8
	3	32 11 19 1 33 12 20 2	38 40
	4	34 26 3 21 13 35 4 22 14 27	31 7 39 25
	5	28 23 5 15 6 24 29	30 6 24 29
	6	30 40 39 25 38 7 36 37 8 31	5 15 23 28 4
	7		14 22 27 35 3
	8		13 21 26 34
	9		2 12 20
	10		33 1 11 19
	11		32 10 18 9 17
	12		16

ing computations of the **S** matrix for the solution of exterior orientation parameters. However, the GPS algorithm is further illustrated by results of the aforementioned examples.

Examples

Table 2 shows the intermediate level structures generated by the GPS reordering program, using both complete and reduced graph structures of the 4 by 11 regular block of photos with down-strip numbering as shown in Figure 1a. Figure 5a then shows the resulting structures of the **S** matrix, using the complete graph structure of photos (Figure 3a) in the GPS reordering program. The bandwidth and profile of the **S** matrix are reduced from those of the original down-strip numbering as shown in Figure 1c. Yet the resulting **S** matrix is not the desired solution for the photogrammetric block. While using the reduced graph structure (Figure 3b) in the GPS reordering program, the optimal solution of the **S** matrix is obtained as shown in Figure 5b, which basically is same as Figure 1d for cross-strip numbering of photos in Figure 1b.

Similarly, Table 3 presents the intermediate level structures of the irregular block of photos with down-strip numbering as shown in Figure 2a. Figures 6a and 6b depict the resulting structures of the **S** matrix of the irregular block which were computed in nearly real-time using the complete (Figure 4a) and the reduced (Figure 4b) graph structures of photos, respectively. The example reveals that using the



(a) (b)

Figure 6. Structures of the **S** matrix of the irregular block in Figure 2a using complete and reduced graph structures of photos in the Automatic Photo Reordering Program. (a) Using complete graph structures. (b) Using reduced graph structures.

complete graph structure in automatic photo reordering yields an **S** matrix with a small bandwidth, and using the reduced graph structures delivers an **S** matrix with a small profile. The choice of using either complete or reduced graph structures in the automatic photo reordering program depends on how the **S** matrix will be manipulated, i.e., applying either Cholesky factorization (minimal profile) or recursive partitioning (minimal bandwidth), for the solution of the reduced normal equations. Most importantly, using both graph structures in the GPS reordering program significantly reduces the bandwidth and the profile of the **S** matrix from those of the original in Figure 2c, and efficiently saves the cost and time required in manual photo numbering for an acceptable solution such as the one shown in Figure 2d. Consequently, full advantages are obtained from using topological connectivities among photos in the automatic photo reordering process.

Conclusions

Full advantage of the simultaneous bundle adjustment of large photogrammetric systems can never be obtained without photo reordering to obtain an **S** matrix with near minimal bandwidth and/or profile. The manual photo ordering approach often requires much pre-processing time and cost, yet may not reach an acceptable photo ordering, especially for irregular blocks of photos. By implementing automatic construction of photo connectivities for graph structures and the GPS algorithm for matrix bandwidth/profile reduction, the automatic photo reordering program provides an efficient and economic solution for a near-optimal **S** matrix regardless of the configuration and numbering scheme of photos in photogrammetric bundle adjustment problems. Meanwhile, the automatic reordering approach is also equally applicable to the least-squares adjustment of survey networks, and saves a bundle for minimal storage requirement and computational cost.

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