1. (x), $(x+1)$, and $(x-e^{-x})$ are 3 solutions of the following linear ordinary differential equation

$$y''+a(x)y'+b(x)y=f(x)$$

Find (1). $a(x), b(x)$, and $f(x)$, (2). the general solution, $y(x)$. (20%)

2. A matrix $A=\begin{bmatrix}-1 & 2 & a\\ 2 & 4 & b\\ 2 & 1 & c\end{bmatrix}$ has two eigenvectors $\phi_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$; $\phi_2=\begin{bmatrix}-1\\0\\2\end{bmatrix}$,

where a, b, c are unknown constants. Find (1). the values of a, b, c, (2). the 3 eigenvalues λ_1, λ_2, λ_3, and (3). the third eigenvector ϕ_3. (20%)

3. A system of ordinary differential equation with initial conditions

$$\begin{bmatrix}y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}+\begin{bmatrix}0\\ \delta(t)\end{bmatrix};\quad \begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

where $\delta(t)$ is the Dirac delta function. Find (1). $y_1(t)$, (2). the maximum value of $y_1(t)$. (20%)

4. Use half-range and Fourier expansion methods to expand the equation:

$f(x)=x^2$, $0<x<L$, (1). in a cosine series, (2). in a sine series, and (3). in a Fourier series. (15%)

5. Solve the following partial differential equation:

$\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}$, with two boundary conditions: $u(0, t)=0$ and $u(L, t)=0$;

and two initial conditions: $u(x, 0)=f(x)$ and $u_t(x, 0)=g(x)$. Also, interpret the physical meanings of eigenvalues and eigenfunctions of this equation. (15%)

6. Evaluate

$$\int_{-\infty}^{\infty}\frac{\sin mx}{x(x^2+k^2)}dx \quad (m\geq 0,\ k>0)$$ (10%)